INADMISSIBILITY OF LARGE CLASSES OF SEQUENTIAL TESTS

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Assume observations are from a subclass of a one parameter exponential family whose dominating measure is nonatomic. Consider a one-sided sequential testing problem where null and alternative parameter sets have one common boundary point. Let the risk function be a linear combination of probability of error and expected sample size. Our main result is that a sequential test is inadmissible if its continuation region has unbounded width in terms of the natural sufficient statistic. We apply this result to prove that weight function tests, with weight functions that contain the common boundary point in their support, are inadmissible. Furthermore any obstructive test is inadmissible, where obstructive means that the stopping time for the test does not have a finite moment generating function for some parameter point. Specific tests of the above type are cited.

1. **Introduction and summary.** Consider the one-sided sequential hypothesis testing problem concerned with the natural parameter of an exponential family. There have been a wide variety of tests recommended and studied for such a problem. An obvious suggestion is to use Bayes tests for an appropriate loss function; see, for example, Chernoff (1965) for references. Bayes tests or "Bayes-type" tests form a complete class for many models. See Brown, Cohen, and Strawderman (1980) and Berk, Brown, and Cohen (1981b). Another suggestion is the method of weight functions. Wald (1947) suggested this method for a formulation that incorporates an indifference region separating the null and alternative hypotheses. Wijsman (1979) and others study certain types of weight function tests that yield invariant sequential probability ratio tests (SPRT's); Wijsman (1979) gives many references. Berk (1970) and Berk (1976, page 905), discusses weight function tests when there is no indifference zone. Robbins (1970) considers a formulation without an indifference zone and uses weight function type tests. Weight functions are essentially prior distributions. Since Bayes tests are often difficult or unfeasible to carry out, weight function tests are appealing. They can be carried out easily and they incorporate prior information, as Bayes tests do.

Among other tests recommended are some closed boundary tests including Anderson's (1960) triangular boundary, Schwartz's (1962) asymptotic optimal boundary, and truncated versions of other tests. There are also open boundary tests in addition to the weight function tests mentioned above. These include power one tests, see Robbins (1970, pages 1405, 1408) for references; a test proposed by Darling and Robbins (see Robbins, 1970, page 1404); sequential generalized likelihood ratio tests proposed by Lorden (1973); and a test proposed by Whitehead (1975).

In this paper we study some of the above types of tests from the point of view of admissibility. We assume observations are from an absolutely continuous distribution which belongs to a one parameter exponential family; there are some further minor technical assumptions. In particular our results include observations from a normal

distribution with unknown mean and known variance. Slight modifications will accommodate the scale parameter of gamma distributions. The one-sided hypotheses we consider are contiguous so that the intersection of the closure of null and closure of alternative is a single boundary point. We assume that the risk function is a linear combination of expected sample size and probability of error. Our main result is that any test whose continuation region, in terms of the natural sufficient statistic, has unbounded width is inadmissible. This result is applied to prove that any weight function test whose weight functions are such that their support includes the boundary point is inadmissible. We also prove that many open-ended tests, including the test of Robbins-Darling, some of Lorden's tests, and Whitehead's test, are inadmissible. Tests of power one are trivially inadmissible for the above risk function. All these open-ended tests are for models without an indifference zone. The main result can also be applied to prove more generally that any test which is obstructive is inadmissible. We define an obstructive test as one whose stopping time does not have a finite moment generating function for at least one parameter point in the union of null, alternative, and boundary.

Clearly some discussion is needed regarding the significance and practicality of these results. In this connection it is necessary to discuss (i) the reasonableness of the loss function, (ii) the nature and pitfalls of the concepts of admissibility and inadmissibility, and (iii) the realization that some of the open-ended procedures were devised for formulations that are sometimes not concerned with loss functions of the type studied here.

(i) The risk function which is a linear combination of expected sample size and probability of error is classical in the sense that it was used by Wald (1947), Le Cam (1955), Lehmann (1959), Ferguson (1967), and others in the development of the subject. More importantly it is a prototype, for perhaps, sometimes more realistic risk functions corresponding to loss functions that are linear combinations of cost of sampling and loss due to incorrect terminal decisions. See, for example, Chernoff (1965). The results of this paper easily extend to some of these other loss functions and can probably be extended to others.

(ii) To assert a test is admissible is not saying much since to be admissible a procedure need only be very good at a single parameter point. Some degenerate procedures are admissible. Inadmissibility is a much more compelling property in spite of the fact that there are occasional examples where intuitive and worthwhile statistical procedures are inadmissible.

(iii) The open-ended tests shown to be inadmissible here were derived to satisfy certain properties, that perhaps are not consistent with optimizing the risk function here. This is certainly the case of power one tests. Nevertheless, if the practitioner feels that a linear combination loss function would not be an unreasonable formulation for his problem, then the results of this paper are advising him to be wary of using such a procedure. As such, we feel that the results of this paper yield important advice to the practitioner.

In addition to the inadmissibility results, we prove that for testing a normal mean, many of the inadmissible weight function tests are also obstructive.

In Section 2 we give preliminaries and discuss facts that will be needed to prove inadmissibility and obstructiveness. Inadmissibility results are given in Section 3 and obstructiveness results in Section 4.

2. Preliminaries and facts. Let $X, X_1, X_2, \cdots$ be a sequence of independent and identically distributed random variables. Let $\mathbf{X} = (X_1, X_2, \cdots)$ and let Θ be a subset of the real line. We assume that there is a probability measure μ which dominates the family $\{P_\theta(\cdot), \theta \in \Theta\}$ of probability measures for $\mathbf{X}$ in the following sense: For each $j = 1, 2, \cdots$, over the σ-field generated by $\mathbf{X}_{(j)} = (X_1, X_2, \cdots, X_j)$, the measure P_θ is dominated by μ. Write $f_{\theta j}(\mathbf{x}_{(j)}) = dP_\theta/d\mu$ relative to this σ-field. We further assume that X is distributed according to an exponential family so that its distribution is

$$(2.1) \qquad P_\theta(dx) = \beta(\theta) \exp\{\theta T(x)\}\mu(dx).$$

We assume that the measure μ is a non-atomic probability measure. The sufficient

transitive sequence $S_n = \sum_{j=1}^n T(X_j)$, at stage n, then has distribution

(2.2) $$P_\theta^{(n)}(ds_n) = \beta^n(\theta) \exp(\theta s_n)\gamma^{(n)}(ds_n),$$

where $\gamma^{(n)}$ is the marginal distribution of S_n. We further assume that $\gamma^{(1)}$ is absolutely continuous with respect to Lebesgue measure and $\gamma^{(1)}$ has an interval as its support. The densities corresponding to (2.1) and (2.2) will sometimes be denoted by $f_\theta(x)$ and $f_{\theta n}(s_n)$ respectively. The null hypothesis space is $\Theta_1 \subset \Theta$ and the alternative space is $\Theta_2 \subset \Theta$. We assume $\Theta = \Theta_1 \cup \Theta_2$. The closures of Θ_1 and Θ_2 are denoted by $\bar\Theta_1$ and $\bar\Theta_2$ respectively. Let $\Omega = \bar\Theta_1 \cap \bar\Theta_2$. We assume $\Omega \neq \{\varnothing\}$, the null set, and $\bar\Theta_1 \subset \mathcal{N}, \bar\Theta_2 \subset \mathcal{N}$, where $\mathcal{N}$ is the natural parameter space. Also assume there exists a positive number K such that $[-K, K] \subset \mathcal{N}$. The facts that $f_\theta(x)$ is exponential family and $\gamma^{(1)}$ is absolutely continuous with respect to Lebesgue measure imply that $\sup_{\theta \in \Theta} f_\theta(x) < \infty$ for almost all x; see Barndorff-Nielsen (1978), Corollary 9.4, page 151. In this paper we consider only one-sided tests so that $\bar\Theta_1 = \{\theta \in \Theta : \theta \leq \theta_0\}$, $\bar\Theta_2 = \{\theta \in \Theta : \theta \geq \theta_0\}$, $\Omega = \{\theta_0\}$.

A prior probability measure Γ on $\Theta(\bar\Theta)$ will be represented by a mixture $\pi_1\Gamma_1 + \pi_2\Gamma_2$. Here π_i is the probability that $\theta \in \Theta_i$ and Γ_i is the conditional distribution of $\theta \in \Theta_i$. We write $\Gamma = (\pi_1, \Gamma_1, \Gamma_2)$. We let Γ_{ni} denote the sequence of posterior probability measures on $\Theta_i(\bar\Theta_i)$, given $S_n = s_n$, so that for a measurable set $A \subset \Theta_i(\bar\Theta_i)$,

(2.3) $$\Gamma_{ni}(A) = \int_A f_{\theta n}(s_n)\Gamma_i(d\theta) \Big/ \left\{ \int_{\Theta_i} f_{\theta n}(s_n)\Gamma_i(d\theta) \right\}.$$

The support of a probability measure $\Gamma(\cdot)$ is defined as supp $\Gamma = \cap\{C : \Gamma(C) = 1, C \text{ closed}\}$.

The action space D consists of pairs (n, τ) where n is the value of the stopping time N, and $\tau = 1$ or 2, depending on whether Θ_1 is accepted or rejected. The loss function, denoted by $L(\theta, (n, \tau)) = cn$ if $\theta \in \Theta_\tau$ and $L(\theta, (n, \tau)) = cn + 1$ if $\theta \notin \Theta_\tau$. Here c represents the cost of each individual observation.

Let

(2.4) $$g_{in}(s_n) = \int_{\Theta_i} f_{\theta n}(s_n)\Gamma_i(d\theta) \Big/ \left\{ \pi_1 \int_{\Theta_1} f_{\theta n}(s_n)\Gamma_1(d\theta) + \pi_2 \int_{\Theta_2} f_{\theta n}(s_n)\Gamma_2(d\theta) \right\},$$

so that $\pi_i g_{in}(s_n)$ is the conditional probability that $\theta \in \Theta_1$, given $S_n = s_n$. Also let $\tau_n(s_n) = \min\{\pi_i g_{in}(s_n) : i = 1, 2\}$ and note $\pi_1 g_{1n}(s_n) + \pi_2 g_{2n}(s_n) \equiv 1$. Whenever $\tau_n(s_n) < c$, the Bayes test stops at stage n; see for example, Ferguson (1967, page 316). Let δ_Γ denote the Bayes test with respect to Γ and let N_Γ denote the stopping time of the Bayes test.

Now let $\tilde\Gamma = (\pi_1, \tilde\Gamma_1, \tilde\Gamma_2)$ where $(\tilde\Gamma_1, \tilde\Gamma_2)$ is a pair of σ-finite measures on $\bar\Theta_1$ and $\bar\Theta_2$ respectively. Define a test $\delta(\mathbf{x})$ to be "generalized Bayes at s_1" if among all $\delta(\mathbf{x})$ it minimizes

(2.5)
$$\pi_1 \int_{\bar\Theta_1} \int_{\chi^*} L(\theta, \delta(\mathbf{x}))P_\theta(d\mathbf{x}^*)\beta(\theta)e^{s_1\theta}\tilde\Gamma_1(d\theta) + \pi_2 \int_{\bar\Theta_2} \int_{\chi^*} L(\theta, \delta(\mathbf{x}))P_\theta(d\mathbf{x}^*))\beta(\theta)e^{s_1\theta}\tilde\Gamma_2(d\theta),$$

where $\mathbf{x}^* = (x_2, x_3, \cdots) \in \chi^*$ and recall $s_1 = T(x_1)$.

The following procedures comprise the class of tests $\bar B$: At stage 0 accept, reject, continue, or randomize between these three possibilities. At stage 1, there is an interval (a_1, a_2), $(-\infty \leq a_1 \leq a_2 \leq \infty)$ such that if $s_1 < a_1$ the procedure stops and accepts $H_1 : \theta \in \bar\Theta_1$; if $s_1 > a_2$, the procedure stops and rejects H_1; if $a_1 < s_1 < a_2$, then the procedure is generalized Bayes at s_1 with respect to a distribution $\Gamma = (\pi_1, \tilde\Gamma_1, \tilde\Gamma_2)$, where $\tilde\Gamma_i$, $i = 1, 2$ are σ-finite measures on $\bar\Theta_i$ and

(2.6) $$\int_{\bar\Theta_i} e^{s_1\theta}\beta(\theta)\tilde\Gamma_i(d\theta) < \infty.$$

The model and assumptions of this section enable us to establish the following facts:

FACT 2.1. *For testing Θ_1 vs Θ_2, the class of tests $\bar{B}$ is complete.* Fact 2.1 is proved by using Theorem 5.1 of Brown, Cohen, and Strawderman (BCS) (1980), and Remark 3.6 of Berk, Brown, and Cohen (BBC) (1981b). Theorem 5.1 of BCS (1980) is stated for the case where the cost of the first observation is zero. When the cost of the first observation is $c > 0$, the only difference is that one has an opportunity to stop at stage 0, which is now reflected in the description of the class $\bar{B}$. From BCS (1980) it is seen that tests in $\bar{B}$ are regular limits of sequences of Bayes tests. Suppose that the cost of the first observation is $c > 0$. Then for any sequence of priors $(\pi_{1k}, \Gamma_{1,k}, \Gamma_{2,k})$, if a subsequence of $\{\pi_{1k}\}$ converges to a number less than c or greater than $(1 - c)$ the limiting procedure of the sequence of Bayes tests would stop at time zero. Hence if the limiting procedure in $\bar{B}$ observes X_1 at stage 1, any subsequences of π_{1k} would have to converge to a limit π_1 such that $c \leq \pi_1 \leq (1 - c)$. This explains why in this paper a test which is generalized Bayes at s_1 is with respect to a distribution $\Gamma = (\pi_1, \tilde{\Gamma}_1, \tilde{\Gamma}_2)$ while in BCS (1980) such a test is with respect to a distribution $\tilde{\Gamma} = (\tilde{\Gamma}_1, \tilde{\Gamma}_2)$.

In what follows it is convenient to partition $\bar{B}$ into the following three mutually exclusive sets: Bayes tests, tests which are not Bayes but stop at stage 1 (i.e., $a_1 = a_2$), tests which are neither of these but are generalized Bayes at s_1. Let these latter tests be designated as GB. Also it is more convenient to let $\Gamma = (\pi_1, \Gamma_1, \Gamma_2)$ represent the distribution for which δ_Γ is Bayes or GB. When Γ_1 and/or Γ_2 are not proper we say Γ is a generalized prior distribution. We allow only those Γ that satisfy (2.6). In the remainder of the paper we write prior distribution to represent proper prior and generalized prior unless a distinction is necessary. We remark here that the results of the present paper will also apply to the case where the cost of the first observation is zero.

FACT 2.2. *Every Bayes test and every test which is GB for testing $\bar{\Theta}_1$ vs $\bar{\Theta}_2$ is monotone.* In BCS (1979) Theorem 3.2 and Example 4.1, it is proved that every Bayes test for testing Θ_1 vs Θ_2 is monotone. The proof works without any changes for testing $\bar{\Theta}_1$ vs $\bar{\Theta}_2$ and for tests which are GB for testing $\bar{\Theta}_1$ vs $\bar{\Theta}_2$.

For every prior distribution Γ, let the monotone δ_Γ be described as follows: For each n, if $a^\Gamma(n) < S_n < b^\Gamma(n)$, continue to stage $(n + 1)$; if $S_n \leq a^\Gamma(n)$, stop and accept; if $S_n \geq b^\Gamma(n)$, stop and reject. Also define $A^\Gamma(n) = a^\Gamma(n)/n$, and $B^\Gamma(n) = b^\Gamma(n)/n$.

FACT 2.3. *For testing Θ_1 vs Θ_2, if $\theta_0 \in \text{supp } \Gamma_\iota$, $i = 1, 2$, then the stopping time of the Bayes test or test which is GB is bounded.* Theorem 3.1 of BBC (1981a) proves the result for Bayes tests. The proof however only requires that the posterior distributions, given $S_n = s_n$, $\Gamma_{ni}(\cdot)$, $i = 1, 2$, be probability distributions, which is the case even if Γ is a generalized prior.

FACT 2.4. *For testing Θ_1 vs Θ_2, the stopping time N_Γ for any Bayes test or test which is GB is exponentially bounded for each $\theta' \in \Theta$.* See BBC (1981b) Section 2, for the definition of exponentially bounded stopping time. See Remark 3.6 of that reference for the proof of Fact 2.4 for Bayes tests. Once again the proof only requires that the posterior distributions given $S_n = s_n$, $\Gamma_{ni}(\cdot)$, $i = 1, 2$ be probability distributions, so that Fact 2.4 is true for tests which are GB.

Let G_ι be probability measures on Θ_ι, $i = 1, 2$. Let $\bar{X} = S_n/n$ and let $0 < \pi_1 < 1$. A weight function test for $G = (\pi_1, G_1, G_2)$ is defined in terms of

$$(2.7) \qquad r_n(\bar{x}) = \pi_2 g_{2n}(n\bar{x})/\pi_1 g_{1n}(n\bar{x})$$

$$= \pi_2 \int_{\Theta_2} \beta^n(\theta) \exp n\bar{x}\theta \, G_2(d\theta)/\pi_1 \int_{\Theta_1} \beta^n(\theta) \exp n\bar{x}\theta \, G_1(d\theta),$$

and a pair of constants $A, B, 0 < A < 1 < B < \infty$. The weight function test (G, A, B) is as follows: Stop and accept if $r_n(\bar{x}) \leq A$, stop and reject if $r_n(\bar{x}) \geq B$, continue if $A < r_n(\bar{x}) < B$. Note that $r_n(\bar{x})$ is a strictly increasing function of $\bar{x}$. It follows that the quantities

$$(2.8) \qquad A_n = A_n(G, A), B_n = B_n(G, B) \text{ defined by } r(A_n) = A \text{ and } r(B_n) = B$$

are uniquely defined. If $a(n) = n A_n$ and $b(n) = n B_n$ then the weight function test can be described as follows: For each n, if $a(n) < S_n < b(n)$, continue to stage $(n + 1)$; if $S_n \leq a(n)$, stop and accept; if $S_n \geq b(n)$, stop and reject. The strict monotonicity of $r_n(\bar{x})$ as a function of $\bar{x}$ implies that $a(n) < b(n)$, and so for any finite integer m, there is a set of positive Lebesgue measure and hence positive probability for all distributions in the model, for which the weight function test continues at least to stage $(m + 1)$.

We will need to study properties of weight function tests $G = (\pi_1, G_1, G_2)$ where $G_i, i = 1, 2$ may not be probability measures but G is a generalized prior. Corollary 3.2 of Section 3 will apply to those weight function tests which do not have a bounded stopping time, which is the case when the G_i are proper or G is a generalized prior.

If the stopping time of a sequential test does not have a finite moment generating function for a value of the parameter θ, say $\theta = \theta^*$, then we say that the stopping time is obstructive at θ^*. In this paper, we say that a test is obstructive if there exists a value of θ, say $\theta = \theta^*$, lying in union of the null, alternative, and boundary such that the test is obstructive at θ^*. See Wijsman (1979) for a slightly different definition and for more discussion of the notion of obstructiveness.

3. Inadmissibility.

The main result is given in this section. We prove that for any test in $\bar{B}$, $\lim_{n\to\infty}\{b^\Gamma(n) - a^\Gamma(n)\} < \infty$, for any prior Γ. This implies that any test whose continuation region has unbounded width cannot lie in $\bar{B}$ and hence is inadmissible. The main result will be applied to prove that weight function tests based on weight functions G such that $\theta_0 \in \text{supp } G_i$ are inadmissible. For such weight function tests we show $\lim_{n\to\infty}\{b(n) - a(n)\} = \infty$. The main result will also yield that any obstructive test is inadmissible.

Note in Fact 2.1 that $\bar{B}$ consists of tests which are Bayes or GB for testing $\bar{\Theta}_1$ vs $\bar{\Theta}_2$. Fact 2.3 however, which will be used subsequently, is stated for tests of Θ_1 vs Θ_2. When testing $\bar{\Theta}_1$ vs $\bar{\Theta}_2$, it may be possible that if $\theta_0 \in \text{supp } \Gamma_i$, $i = 1, 2$, that the stopping time is unbounded. The priors with $\theta_0 \in \text{supp } \Gamma_i$ that may result in N_Γ being unbounded, must put point mass at θ_0 as part of $\bar{\Theta}_1$ and as part of $\bar{\Theta}_2$; See BCS (1980), page 382 for clarification of this specification. Furthermore, there must exist a number $\theta_1 < \theta_0$ and/or a number $\theta_2 > \theta_0$ such that Γ_1 puts no mass on the interval (θ_1, θ_0) and/or Γ_2 puts no mass on the interval (θ_0, θ_2), while θ_1 and/or θ_2 lies in the support of Γ_1 and/or Γ_2 respectively. Call such priors, gap priors. For all nongap priors with $\theta_0 \in \text{supp } \Gamma_i$, $i = 1, 2$, N_Γ has bounded stopping time; see the arguments in BBC (1981a). Keep in mind that the weight function tests to be shown inadmissible are tests for Θ_1 vs Θ_2. Recall from Fact 2.1 that $\bar{B}$ is a complete class for testing Θ_1 vs Θ_2 but that tests in $\bar{B}$ are tests for $\bar{\Theta}_1$ vs $\bar{\Theta}_2$. $\bar{B}$ is also a complete class for testing $\bar{\Theta}_1$ vs $\bar{\Theta}_2$; see BCS (1980). When we refer to Bayes or GB tests we regard them as tests of $\bar{\Theta}_1$ vs $\bar{\Theta}_2$. It will be necessary to study weight functions tests related to Bayes and GB tests. That is, in the proofs of the lemmas to follow we need to study properties of weight function tests with the same prior as the given Bayes or GB test. At such times these related weight function tests are for $\bar{\Theta}_1$ vs $\bar{\Theta}_2$. As the proofs are developed the above remarks will be elucidated.

LEMMA 3.1. *Let Γ be a prior on $\bar{\Theta}_1 \cup \bar{\Theta}_2$. Then*

$$(3.1) \qquad A^\Gamma(n) \geq A_n(\Gamma, [c/(1 - c)]), B^\Gamma(n) \leq B_n(\Gamma, [(1 - c)/c]),$$

where A_n and B_n are defined in (2.8).

PROOF.   Recall that the Bayes or GB test (hereafter we shall just write Bayes test) will stop if $\tau_n(s_n) \leq c$. In other words the Bayes test will stop whenever $\pi_i g_{in}(s_n) \leq c$, for $i = 1$ or 2. From (2.4) and (2.7) this will occur whenever $r_n(\bar{x}) \geq (1 - c)/c$ or $r_n(\bar{x}) \leq c/(1 - c)$. □

Now let $\theta' = \sup\{\theta : \theta \in \text{supp } \Gamma_1\}$, $\theta'' = \inf\{\theta : \theta \in \text{supp } \Gamma_2\}$, and $t^* = \{\log \beta(\theta') - \log \beta(\theta'')\}/(\theta'' - \theta')$. Note that θ^* defined by $E_{\theta^*} = t^*$ is equidistant from θ' and θ'' in the Kullback-Liebler sense. Also assume $-\infty < \theta' \leq \theta'' < \infty$.

LEMMA 3.2.   *Let Γ be a prior on $\bar{\Theta}_1 \cup \bar{\Theta}_2$, but not a gap prior. Let A, B be any pair of fixed numbers such that $0 < A < 1 < B < \infty$. Then*

$$(3.2) \quad \lim_{n\to\infty} A_n(\Gamma, A) = t^*, \; \lim_{n\to\infty} A^\Gamma(n) = t^*, \; \lim_{n\to\infty} B_n(\Gamma, B) = t^*, \; \lim_{n\to\infty} B^\Gamma(n) = t^*.$$

PROOF.   Consider $r_n(t)$ defined in (2.7). Recall $r_n(t)$ is strictly increasing in t. For any fixed t such that $t^* < t$ or $t^* > t$, it follows using the definition of t^*, that $r_n(t) \to \infty$ or 0 respectively as $n \to \infty$. This yields the desired conclusion for A_n and B_n. Use Lemma 3.1 to obtain the conclusion for $A^\Gamma(n)$ and $B^\Gamma(n)$. □

LEMMA 3.3.   *Let Γ be a prior on $\bar{\Theta}_1 \cup \bar{\Theta}_2$, but not a gap prior. Let A, B be as in Lemma 3.2. If $\theta'' - \theta' > 0$, then*

$$(3.3) \quad \lim_{n\to\infty} n\{B^\Gamma(n) - A^\Gamma(n)\} < \infty \text{ and } \lim_{n\to\infty} n\{B_n(\Gamma, B) - A_n(\Gamma, A)\} < \infty.$$

If $\theta'' = \theta' = \theta_0$, then

$$(3.4) \quad \lim_{n\to\infty} n\{B_n(\Gamma, B) - A_n(\Gamma, A)\} = \infty.$$

PROOF.   Let $\lambda_n(t) = \log r_n(t)$ where $r_n(t)$ is defined in (2.7). Note that

$$(3.5) \quad (\partial/\partial t)\lambda_n(t) = n\left\{\int_{\theta''}^{\infty} \theta e^{n\theta t}\beta^n(\theta)\Gamma_2(d\theta) \Big/ \int_{\theta''}^{\infty} e^{n\theta t}\beta^n(\theta)\Gamma_2(d\theta)\right.$$
$$\left. - \int_{-\infty}^{\theta'} \theta e^{n\theta t}\beta^n(\theta)\Gamma_1(d\theta) \Big/ \int_{-\infty}^{\theta'} e^{n\theta t}\beta^n(\theta)\Gamma_1(d\theta)\right\}.$$

By Lemma 3.2 we need only be concerned with values of t in a small neighborhood of $t = t^*$. Uniformly for such values

$$\int_{\theta''}^{\infty} \theta e^{n\theta t}\beta^n(\theta)\Gamma_2(d\theta) \Big/ \int_{\theta''}^{\infty} e^{n\theta t}\beta^n(\theta)\Gamma_2(d\theta) \to \theta''$$

and

$$\int_{-\infty}^{\theta'} \theta e^{n\theta t}\beta^n(\theta)\Gamma_1(d\theta) \Big/ \int_{-\infty}^{\theta'} e^{n\theta t}\beta^n(\theta)\Gamma_1(d\theta) \to \theta'.$$

Hence $(\partial/\partial t)\lambda_n(t) \to n(\theta'' - \theta')$, uniformly for all t in a neighborhood of t^*. It follows by use of (3.2) and the mean value theorem that

$$(3.6) \quad \{\lambda_n(B_n(\Gamma, B)) - \lambda_n(A_n(\Gamma, A))\}/\{B_n(\Gamma, B) - A_n(\Gamma, A)\} = (\partial/\partial t)\lambda_n(t)\big|_{t=t'},$$

where t' is in the neighborhood of t^*. Thus, from (3.5) and (3.6) we have

$$(3.7) \quad \lim_{n\to\infty} n\{B_n(\Gamma, B) - A_n(\Gamma, A)\} = \{(\log B - \log A)/(\theta'' - \theta')\} < \infty$$

if $\theta'' - \theta' > 0$. Lemma 3.1 yields the other statement in (3.3). If $\theta'' = \theta' = \theta_0$ then given $\varepsilon > 0$, ε arbitrarily small, there exists a neighborhood about t_0, where $E_{\theta_0} X = t_0$, such that

uniformly for all t in the neighborhood, as $n \to \infty$, the lim sup of the bracketed expression in (3.5) is bounded above by 2ε. Thus, in this case $\lim_{n\to\infty} n\{B_n(\Gamma, B) - A_n(\Gamma, A)\} \geq (\log B - \log A)/2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\lim_{n\to\infty} n\{B_n(\Gamma, B) - A_n(\Gamma, A)\} = \infty$. □

LEMMA 3.4. *Let Γ be a gap prior on $\bar{\Theta}_1 \cup \bar{\Theta}_2$. Then $\lim_{n\to\infty} \{b^{\Gamma}(n) - a^{\Gamma}(n)\} < \infty$.*

PROOF. The Bayes or GB test stops at stage n whenever $\pi_i g_{in}(s_n) \leq c$ for $i = 1$ or 2. Lemma 2.2 of BBC (1981a) implies that the Bayes or GB test will also stop at stage n if for $i = 1$ and 2, $\Gamma_{ni}(\{\theta_0\}) > 1 - \varepsilon$, for a suitably chosen small $\varepsilon > 0$. Hence let us study $\pi_i g_{in}(s_n)$ and $\Gamma_{ni}(\{\theta_0\})$. Note that

(3.8)
$$\pi_2 g_{2n}(t) = \left\{ \pi_2 \gamma_2 \beta^n(\theta_0) e^{nt\theta_0} + \pi_2 \int_{\theta_2}^{\infty} \beta^n(\theta) e^{nt\theta} \Gamma_2(d\theta) \right\} \bigg/$$
$$\left\{ \pi_1 \gamma_1 \beta^n(\theta_0) e^{nt\theta_0} + \pi_1 \int_{-\infty}^{\theta_1} \beta^n(\theta) e^{nt\theta} \Gamma_1(d\theta) \right.$$
$$\left. + \pi_2 \gamma_2 \beta^n(\theta_0) e^{nt\theta_0} + \pi_2 \int_{\theta_2}^{\infty} \beta^n(\theta) e^{nt\theta} \Gamma_2(d\theta) \right\}$$
$$= (\pi_2 \gamma_2 + u_2(t))/\{\pi_1 \gamma_1 + u_1(t) + \pi_2 \gamma_2 + u_2(t)\},$$

where $0 < \gamma_i < 1$, $i = 1, 2$, and

(3.9)
$$u_2(t) = \pi_2 \int_{\theta_2}^{\infty} \beta^n(\theta) e^{nt\theta} \Gamma_2(d\theta) / \beta^n(\theta_0) e^{nt\theta_0},$$
$$u_1(t) = \pi_1 \int_{-\infty}^{\theta_1} \beta^n(\theta) e^{nt\theta} \Gamma_1(d\theta) / \beta^n(\theta_0) e^{nt\theta_0}.$$

Also note that

(3.10)
$$\Gamma_{ni}(\{\theta_0\}) = \pi_i \gamma_i / \{\pi_i \gamma_i + u_i(t)\}.$$

Now let $\tilde{t}_i$ be the point such that $E_{\tilde{\theta}_i} X = \tilde{t}_i$, and $\tilde{\theta}_i$ is equidistant in the Kullback-Liebler sense from θ_0 and θ_i, $i = 1, 2$. That is, for example, $\tilde{t}_2 = \{\log \beta(\theta_0) - \log \beta(\theta_2)\}/(\theta_2 - \theta_0)$. Recognize that $u_2(t)$ has the properties of $r_n(t)$ in the sense that it is strictly increasing in t and tends to ∞ as $t \to \infty$. Furthermore for fixed $t > \tilde{t}_2$, $\lim_{n\to\infty} u_2(t) = \infty$ and for fixed $t < \tilde{t}_2$ $\lim_{n\to\infty} u_2(t) = 0$. The properties of $\{u_1(t)\}^{-1}$ can be similarly expressed. Furthermore for fixed $t > t_0$, where $t_0 = E_{\theta_0} X$, $\lim_{n\to\infty} u_1(t) = 0$, while for fixed $t < t_0$, $\lim_{n\to\infty} u_2(t) = 0$. These facts imply that for n sufficiently large the event $u_2(t) > R_{22}$ is contained in the event $\pi_1 g_{1n}(t) \leq c$, the event $u_2(t) < R_{12}$ is contained in the event $\Gamma_{n2}(\{\theta_0\}) > 1 - \varepsilon$, the event $u_1(t) > R_{21}$ is contained in the event $\pi_2 g_{2n}(t) \leq c$, and the event $u_1(t) < R_{11}$ is contained in the event $\Gamma_{n1}(\{\theta_0\}) > 1 - \varepsilon$, where R_{ij}, $i = 1, 2$; $j = 1, 2$ are constants such that $0 < R_{11} \leq R_{21} < 1$; $0 < R_{12} \leq R_{22} < 1$. For $t > t_0$, and $u_2(t) < R_{12}$ and n sufficiently large, $\Gamma_{n2}(\{\theta_0\}) > 1 - \varepsilon$ and $\Gamma_{n1}(\{\theta_0\}) > 1 - \varepsilon$ also, since for such t, $u_1(t)$ will be close to zero. Similarly for $u_1(t) < R_{11}$ with $t < t_0$. Thus the $u_i(t)$ and R_{ij} determine a pair of stopping boundaries, call them $U_{ij}(n)$, such that for all $n > m$ (m sufficiently large) the Bayes or GB test stops at least as often as t lies outside the intervals $[U_{11}(n), U_{21}(n)]$, $[U_{12}(n), U_{22}(n)]$. Treating $u_2(t)$ as we did $r_n(t)$ in Lemma 3.3, treating $\tilde{t}_2$ as t^*, θ_2 as θ'', θ_0 as θ', using the same argument as in Lemma 3.3 we find

(3.11) $\lim_{n\to\infty} n\{U_{22}(n) - U_{12}(n)\} < \infty$ and $\lim_{n\to\infty} n\{U_{21}(n) - U_{11}(n)\} < \infty$.

Since the Bayes or GB test is monotone by Fact 2.2 the stopping bounds for δ_{Γ} are

contained within either $[U_{11}(n), U_{21}(n)]$ or $[U_{12}(n), U_{22}(n)]$ for sufficiently large n. It follows then from (3.11) that $\lim_{n\to\infty}(b_n^{\Gamma} - a_n^{\Gamma}) < \infty$. $\square$

A remark pertaining to Lemma 3.4 will be made after Corollary 3.2 below. Now we prove

THEOREM 3.1. *Any test whose continuation region has unbounded width is inadmissible.*

PROOF. A complete class of tests is $\bar{B}$, where $\bar{B}$ is defined in Section 2 before Fact 2.1. Tests which stop at stage 1 have bounded stopping time. All other tests in $\bar{B}$ are Bayes or GB. Suppose $H = (\pi_1^H, H_1, H_2)$ is a non-gap prior with θ_0 lying in the support of H_ι. Then it follows from Theorem 3.1 of BBC (1981a) that δ_H has bounded stopping time. Hence for such priors $\lim_{n\to\infty}n(B_n^H - A_n^H) = 0$. For priors not satisfying the condition that θ_0 lies in the support of both H_ι, Lemma 3.3, equation (3.3) implies $\lim n(B_n^H - A_n^H) < \infty$, while for gap priors Lemma 3.4 implies $\lim n(B_n^H - A_n^H) < \infty$. This covers all tests in $\bar{B}$. Thus all tests in $\bar{B}$ have continuation regions with bounded width. $\square$

COROLLARY 3.2. *Let $\Gamma = (\pi_1, \Gamma_1, \Gamma_2)$ be a prior on Θ such that θ_0 is in the support of both Γ_1 and Γ_2. Then any weight function test (Γ, A, B), $0 < A < 1 < B$, is inadmissible.*

PROOF. For the given weight function test, equation (3.4) states that $\lim_{n\to\infty}n\{B_n(\Gamma, B) - A_n(\Gamma, A)\} = \infty$. The corollary now follows immediately from Theorem 3.1. $\square$

REMARK 3.1. Lemma 3.4 was used to prove that (Γ, A, B) could not be Bayes or GB with respect to a gap prior. This fact can be proved more easily than in Lemma 3.4 but the easier proof does not give the property stated in Lemma 3.4. The easier proof is as follows: Without loss of generality let $t_0 = 0$. There exists as $\varepsilon > 0$ such that for all n sufficiently large, the Bayes or GB test with respect to a gap prior, will stop at stage n if $|\bar{X}_n| < \varepsilon$. This follows from Lemma 2.2 of BBC (1981a) and the beginning of the proof of Theorem 3.1 of that reference. Hence the stopping region for large n includes the set $-n\varepsilon < S_n < n\varepsilon$. On the other hand, from Lemmas 3.2 and 3.3 we have $A_n(\Gamma, A) \to 0$, $B_n(\Gamma, B) \to 0$ and either $n B_n(\Gamma, B) \to \infty$ or $n A_n(\Gamma, A) \to -\infty$. Hence the continuation region for the weight function test must include some of the stopping region for the Bayes or GB test.

COROLLARY 3.3 *Any test which is obstructive is inadmissible.*

PROOF. Theorem 3.1 implies that for any test in $\bar{B}$, $\lim\{b^{\Gamma}(n) - a^{\Gamma}(n)\} < \infty$. It follows as a consequence of Stein's Lemma, see Wijsman, 1979, page 249, that any such test has exponentially bounded stopping time for all $\theta \in \bar{\Theta}_1 \cup \bar{\Theta}_2$. Thus any obstructive test is inadmissible. $\square$

The test of Darling-Robbins, Lorden's tests of contiguous hypotheses, and Whitehead's test are all easily seen to be obstructive. This is so because for each test there exists at least one parameter point in the union of null, alternative, or boundary for which the expected stopping time is infinite. Tests of power one are obviously obstructive and therefore inadmissible. However, this result does not require Corollary 3.3, since the test which rejects without any observations is better than a power one test for the linear combination loss function.

4. Obstructiveness. In this section we assume X is normal with mean θ and known variance σ^2, which without loss of generality is taken to be 1. Also θ_0 is without loss of generality set equal to 0. Let (G, A, B) be a weight function test and we claim that most

of the (G, A, B) for which $\{0\}$ lies in the support of G_i, $i = 1, 2$, are obstructive at 0. More precisely obstructiveness at 0 could be established for the following four nonmutually exclusive cases:

(i) Symmetric case: $\pi_1 = \pi_2$, $G_1(-d\theta) = G_2(d\theta)$
(ii) Partly symmetric case: $\pi_1 \neq \pi_2$, $G_1(-d\theta) = G_2(d\theta)$, with a condition on π_1.

Let $G_i(d\theta) = g_i(\theta)\, d\theta$, $i = 1, 2$. That is, $g_i(\theta)$ are densities.

(iii) $\lim_{\theta \to 0^-} g_1(\theta) = \gamma_1 > 0$, $\lim_{\theta \to 0^+} g_2(\theta) = \gamma_2 > 0$.
(iv) $\lim_{\theta \to 0^-} g_1(\theta) = 0$ and/or $\lim_{\theta \to 0^+} g_2(\theta) = 0$, and $0 < \lim_{\theta \to 0^-} g_1^{(k_1)}(\theta) < \infty$ and $0 < \lim_{\theta \to 0^+} g_2^{(k_2)}(\theta) < \infty$, where $g_1^{(k_1)}$ is the k_1th derivative of $g_1(\theta)$ and k_1 is the smallest positive integer for which $0 < \lim_{\theta \to 0^-} g_1^{(k_1)}(\theta) < \infty$. Similarly for k_2 and $g_2^{(k_2)}$.

For proving cases (i) and (ii), a theorem of Lai (1977, pages 211–212) could be applied. For the other cases, a theorem of Lai and Wijsman (1979, pages 675–676), could be applied. We omit the details.

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